capstract:2611.00001v1 [hep-th] 2 Nov 2026

Boundary Factorization with a Partially Recorded Center

Leona V. Haldersen[1,*] Masaki Tsubehara[2] Nayana Selvaratne[2,3]
[1]Department of Mathematical Physics, Ravelin Institute, Bergen, Norway
[2]Centre for Quantum Geometry, Kosenbridge Institute of Science, Sendai, Japan
[3]Department of Theoretical Physics, Haldenmere College, Wellington, New Zealand

November 2, 2026

Abstract

A finite semisimple boundary algebra gives an exact model of the distinction between sewing at fixed central character and averaging over central characters before testing boundary factorization. We derive the joint amplitudes of classical boundary readouts coupled through this center and give a rank bound on the number of sectors compatible with the readouts. In an eight-sector realization the bound is saturated even though each single-boundary marginal agrees with that of the independently prepared system. A charge selection rule controls higher moments and separates the shared preparation from a mixture that agrees on all pair observables. Conditioning on a partial record of the sector lowers the rank of the defect, with full factorization recovered only when the record distinguishes all occupied sectors. We instantiate the readout using a certification procedure for an absent original and distinguish removal of the original from removal of the shared preparation. All results follow from finite sums, including the comparison between empty and partially specified boundary records.

Keywords: topological field theory, factorization, superselection sectors, boundary algebras

1 Introduction

The product of partition functions on two boundaries in a specified theory and the mean of that product over theories are different objects [7, 11, 16]. In JT gravity the difference has a geometrical interpretation through the double trumpet in the matrix integral [18]. The Lorentzian obstruction involving the bulk Hilbert space [12] raises a related question with different observables. Euclidean examples with dynamical metrics depend on which data are integrated over in common [4, 17].

Given a joint readout at two boundaries, how many sectors must a conditionally independent description have? We give an exact bound and calculate what remains of the factorization defect when a partial record of the sector is available. A two-dimensional topological field theory identifies the sewn amplitudes that reproduce these moments. The normalization is specified in Section 2.

The example came from a determination that two intake points could independently certify the nonreceipt of a requisition [2]. Each used a complete local record. Instructions prohibited either point from seeing the other's copy. The absence of a submission was taken to exclude any

∗Corresponding author: l.haldersen@ravelin.example

common input. The registration mark on the certificate stock had, however, been impressed before either point opened. The requisition was never made.

2 The circle algebra and the boundary state

Let $Q = \mathbb{Z}/q\mathbb{Z}$, with $q \geq 2$. Take the oriented circle algebra to be $\mathcal{A} = \mathbb{C}^Q$ with pointwise multiplication and identity $\mathbf{1}$. Write e_a for the idempotent supported on $a \in Q$. Our Frobenius trace and the positive form it induces are specified in Eq. (1). This is the semisimple case of the oriented bordism construction [1, 3, 15].

$$\tau(e_a) = q^{-1}, \qquad (f, g)_\tau = \tau(\overline{f} g). \tag{1}$$

The orthonormal charge basis $\chi_m(a) = \exp(2\pi i m a/q)$ identifies $\mathcal{A}$ with the complex group algebra of Q in its normalized coefficient trace. Related character sums occur in finite gauge fields [8] and two-dimensional Yang–Mills at zero area [21]. We choose Eq. (1) directly, since a different gauge group volume would change the bare handle weights.

At a pair of pants, sector labels on the circles must coincide. The inverse metric on a sewn circle is $q\delta_{ab}$, so a connected genus-zero bordism with n circle states gives $\tau(f_1 \cdots f_n)$. We pair all outgoing circles with the displayed states, inserting the complex conjugate of any state regarded as a bra.

Let Φ_{sh} describe n boundary units prepared in one shared unknown sector. In the split state Φ_{sp} each unit uses its own draw. Both functionals in Eq. (2) are positive and normalized on $\mathcal{A}^{\otimes n}$. Setting all but one insertion to the identity gives equal marginals.

$$\begin{aligned} \phi_a(f) &= f(a), \\ \Phi_{\mathrm{sh}}(f_1 \otimes \cdots \otimes f_n) &= \frac{1}{q} \sum_{a \in Q} \prod_{i=1}^{n} \phi_a(f_i), \\ \Phi_{\mathrm{sp}}(f_1 \otimes \cdots \otimes f_n) &= \prod_{i=1}^{n} \tau(f_i). \end{aligned} \tag{2}$$

The connected genus-zero amplitude gives Φ_{sh}. For n disjoint disks the functor gives Φ_{sp}. A connected expectation in the statistical sense subtracts products of lower moments. In particular its two-boundary value is the difference of the two functionals. The connected geometry itself also includes the constant mode. Appendix A gives the handle normalization.

Equivalently one can take sector states in a diagonal Hilbert space, with the boundary operators acting by multiplication. A uniform incoherent mixture and a vector with component norm squared q^{-1} in each sector give the same expectations of these operators. Gauge invariant algebras can acquire a common center by imposing a constraint on the physical Hilbert space [5, 9]. Here the center comes from record preparation.

3 Realization by independent certification

3.1 Local records and the common preparation

A determination of nonreceipt states that no matching requisition was entered and identifies the register row in which the search ended. Both fields are completed even if the register is empty.

The missing document is a request for a replacement blank. Its instance number would have been assigned at intake, so the local search uses the form type and the intake window. We use the definitions of same and separate determinations in [2].

The service copy stock has eight alternatives. Alternative a offsets the register alphabet by a relative to the reference column on the backing sheet. A common batch gives each point a fresh sheet prepared with the same setting. In the split case each sheet has its own setting. The visible row b gives no indication of whether the shift was used. The backing sheet records a and is filed under stock inspection, outside the disposition register [14]. After dispatch the readers work independently.

A local key directs the reader at point i to the main setting with fraction λ_i. The remaining keys are divided equally among the q reference rows. A reference row can coincide with the main setting, which is why Eq. (3) includes its uniform term on the diagonal as well. The fractions are design parameters. A routine log in which readers choose keys after inspecting the backing sheet would have a different channel.

$$R_i(b \mid a) = \lambda_i \delta_{ba} + \frac{1-\lambda_i}{q}, \qquad 0 \le \lambda_i \le 1. \tag{3}$$

In the nominal enumeration each point has 24 main and eight distinct reference keys. We count each combination of keys once for every setting of the backing sheet. For the split case a separate setting is counted for each point. These are exact dictionaries. A row of the dictionary is a permitted service record and carries no claim of a separate delivery.

The third and fourth intake points are appeal registers. An appeal asks whether the original determination should stand and is eligible even if the requisition was never made. We hold the appeal disposition fixed and record only its row. The appeal copies use the same batch and different local keys. Attaching a scan of an earlier determination would change the joint law [20].

3.2 Insertions induced by the reader

For a function u of the observed row, the inserted state is its conditional mean. Eq. (4) preserves the constant mode and attenuates every nonzero mode by the same fraction.

$$(R_i^{*}u)(a) = \sum_{b \in Q} R_i(b \mid a)\, u(b), \qquad R_i^{*}\chi_m = \begin{cases} 1 & m = 0, \\ \lambda_i \chi_m & m \neq 0. \end{cases} \tag{4}$$

We grant access to the backing sheet only in Section 5, after all dispositions are fixed. Keeping it out of the local observable algebra makes the number of sectors an inference from the service copies.

4 Observable signatures of a common sector

4.1 Pair amplitudes and a lower bound on the center

Let J_{xy} be the joint probability of rows x, y at the two points. Evaluate Eq. (2) on the corresponding indicators and write $P = q^{-1}\mathbf{1}\mathbf{1}^{\mathsf{T}}$, $\Lambda = \lambda_L \lambda_R$. Each marginal is uniform. The probability of equal rows is $q^{-1}[1 + (q-1)\Lambda]$ in the shared state.

$$J_{\mathrm{sh}} = q^{-1}\big[P + \Lambda(I - P)\big],$$
$$J_{\mathrm{sp}} = q^{-1}P. \tag{5}$$

In Figure 1 the diagonal probability is $79/1024$ per cell and the off-diagonal is $7/1024$. Table 1 compares the split preparation to a right reader switched to reference keys. Both controls give the same probability matrix. The reference control keeps the common setting, which would become observable again if the reader resumed its main-key schedule.

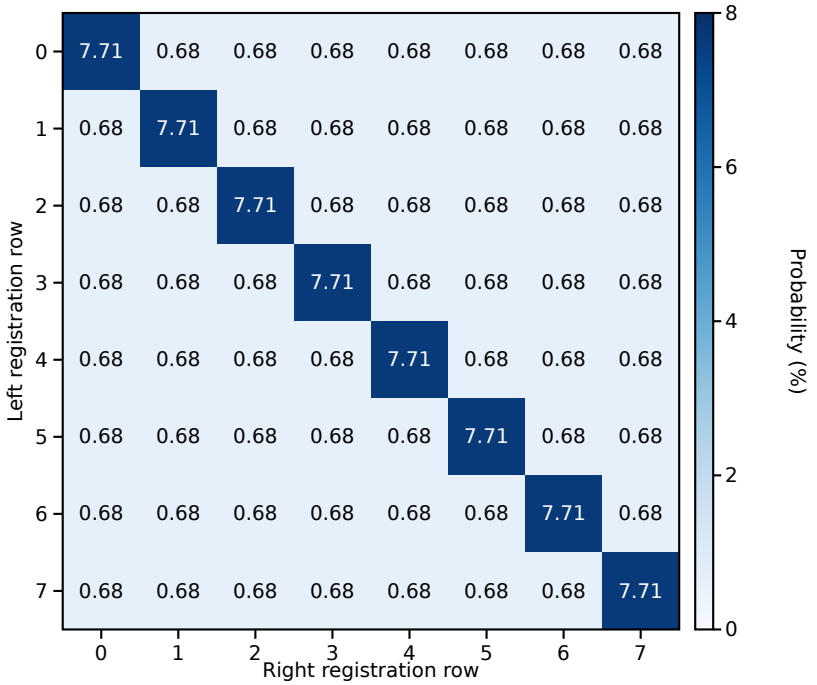


Figure 1: Joint probability by registration row, in percent, for the shared preparation with q = 8 and main-key fraction 3/4 at both points. Each point separately has uniform row probability.

Table 1: Pair laws for nominal and control preparations. The final column gives the singular value of the centered probability matrix on its nonconstant subspace. A reference-only reader uses eight keys, one per row.

Preparation	Dictionary entries	Equal rows	Probability of equal rows	Singular value
Shared, two nominal readers	8,192	5,056	$79/128$	$9/128$
Split, two nominal readers	65,536	8,192	$1/8$	0
Shared, right reader reference only	2,048	256	$1/8$	0

In Eq. (5) the centered matrix has rank $q - 1$ when both readers sometimes use the main key. Before centering it has rank q. A mixture with s sectors has $J = \sum_{a=1}^{s} p_a u_a v_a^{\mathsf{T}}$ for stochastic columns u_a, v_a. Thus $s \geq \mathrm{rank}(J)$ [13]. The present construction attains the bound. A binary dis-

tinction such as whether a blank was requested cannot account for Figure 1 if it exhausts what the readers share.

For the uncentered shared matrix, a perturbation of operator norm less than Λ/q cannot reduce rank. The exact split result has no such stability. Even a finite count table from independently sampled sheets will usually have rank greater than one. An archival test would require a sampling model and a tolerance for weak singular values.

The mutual information is 0.9653 bits. Weight $\log_2(qp)$ by p and $\log_2[q(1-p)/(q-1)]$ by $1-p$, with $p = 79/128$ the equality probability [19]. The mutual information tends to zero continuously as Λ tends to zero, while the eight-sector rank bound holds for every $\Lambda > 0$.

4.2 Higher boundary observables

A simultaneous translation of all visible rows multiplies a character product by the phase of its total charge. Averaging over the common setting projects onto neutral products, as in Eq. (6). Zero-charge insertions are identities. In the split state any nonzero charge makes the moment vanish.

$$\mathbb{E}_{\mathrm{sh}}\left[\prod_{i=1}^{n}\chi_{m_i}(B_i)\right] = \mathbf{1}_{\sum_i m_i=0 \bmod q}\prod_{i:m_i\neq 0}\lambda_i. \tag{6}$$

For a neutral tuple with no proper neutral subtuple, each nontrivial partition contains a block of zero moment. Its full moment is its cumulant. Examples at $q=8$ are $(1,7)$, $(1,2,5)$ and $(1,1,1,5)$. At $\lambda = 3/4$ the third- and fourth-order predictions in Figure 2 are $27/64$ and $81/256$. The main-key fraction is unchanged at appeal.

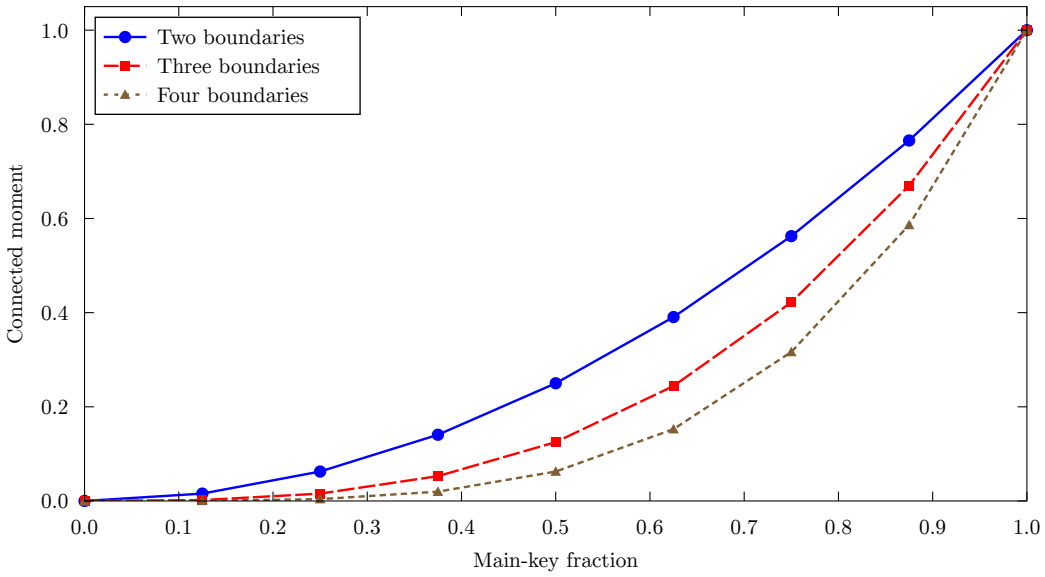


Figure 2: Connected moments for three neutral charge tuples with no proper neutral subtuple. Each marker is an exact enumeration on the 64-key schedule. Lines connect the enumerated points.

At $\lambda = 1/2$, select one of the three pairs with probability $1/4$ each. Assign that pair a common uniform row and the third reader an independent one. With the remaining probability $1/4$ use three independent rows. Each pair has the mixture of a uniform equal-row law of weight $1/4$ and an independent law of weight $3/4$, in agreement with Eq. (5). The $(1, 2, 5)$ moment vanishes because there is always an independent nonzero character. For a shared sector it is $1/8$.

At $3/4$ those pair weights would sum to more than one. The ambiguity persists. Add the real part of the $(1, 2, 5)$ character product times δ/q^3 to the shared three-row law. Summing over any deleted row kills the added term. The all-reference contribution keeps the law positive for $|\delta| < (1 - \lambda)^3$. The triple moment shifts by $\delta/2$ since the double of the tuple is nonzero in Q^3. Even full pair data leave the appeal amplitude undetermined absent a preparation rule.

5 Conditioning on the supporting copy

5.1 Partial records

Suppose the abstract from the backing sheet is $T_d = a \bmod d$, with d dividing q. The available records are zero, one, two or three low bits. Within stratum t, the sector is uniform on $A_t = \{t + jd : 0 \le j < q/d\}$. We test independence within each stratum. Pooling over the strata recovers Eq. (2).

Write p_t for the probability vector of the uniform law on A_t. The reference terms cancel in the centered cross-covariance K_t of row indicators. A row outside A_t can still appear through a local reference key, but its conditional probability is independent of the sector. Eq. (7) follows by taking the covariance of the conditional mean vectors.

$$K_t = \Lambda\big[\operatorname{diag}(p_t) - p_t p_t^{\mathsf{T}}\big], \qquad \operatorname{rank}(K_t) = q/d - 1 \quad (\Lambda > 0). \tag{7}$$

The nonzero singular values at fixed d are $\Lambda d/q$. Each increases as the stratum narrows, even though fewer directions remain correlated. The trace decreases as $\Lambda(1 - d/q)$. To obtain the diagonal character covariance in Eq. (8), use that the sector mean of χ_m has modulus one for multiples of q/d and zero for other charges.

$$\operatorname{Cov}\Big(\chi_m(B_L), \overline{\chi_m(B_R)} \mid T_d = t\Big) = \Lambda\left(1 - \mathbf{1}_{m=0 \bmod (q/d)}\right), \qquad m \neq 0. \tag{8}$$

Figure 3 and Table 2 give the conditional results. At $d = 4$ four nonzero charge probes still have nonzero diagonal entries. Restricted to the two sectors in a stratum, their centered values span just one direction. If instead one averages the centered matrices across strata, their supports are orthogonal, so the rank becomes $q - d$ at $\Lambda > 0$ and the nonzero eigenvalues are Λ/q.

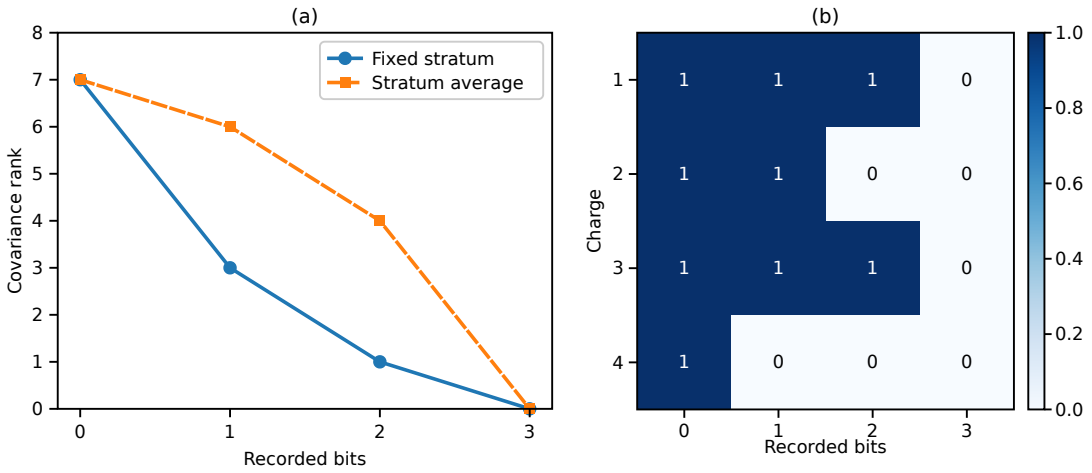


Figure 3: Conditional factorization defect with nominal readout. Left, rank within each stratum and rank after averaging the centered matrices. Right, the diagonal covariance in four charge modes, normalized by the unconditioned nonzero-mode covariance.

Table 2: Exact conditional results for the common preparation with two nominal readers. The number of possible sectors and the entropy refer to a single observed value of the supporting record. The trace is given in row-indicator coordinates.

Recorded bits	Sectors compatible	Entropy (bits)	Rank within stratum	Conditional trace
0	8	3	7	$63/128$
1	4	2	3	$27/64$
2	2	1	1	$9/32$
3	1	0	0	0

5.2 Removal of the originating field

The nonreceipt procedure records a withdrawal on both supporting copies [6]. In our model its designation is fixed before choosing the backing sheet. It changes a deterministic disposition and leaves Eq. (5) and Eq. (6) unchanged. If the originating number depended on the setting, removing it would also remove potential sector information.

Filing the disposition alone replaces the insertion by the identity, killing every registration cumulant of order at least two involving that leg. Issuing fresh stock independently at each point gives the split state. Filing the full alignment mark gives a fixed-sector test. The fixed-sector marginals put extra mass on the observed setting, while the unstratified split marginals are uniform.

The abstractor's manual calls for an empty registration column if the backing sheet was returned unexamined [14]. This has $d = 1$. A zero written in a completed three-bit field has $d = 8$ and selects $a = 0$. If the database imports both as numerical zero, it can mistake a rank-seven mixture for a factorizing stratum. Every setting generates the same number of certificates, so the count of issued certificates cannot expose this error.

6 Scope of the correspondence

The rank bound assumes a mixture of local products. Direct interaction could fit the same probability matrix in a one-sector model if conditional independence failed. We rule out that interaction after dispatch by construction. Observational records from independent mailings would leave the cause of covariance unidentified. The comparison with separately prepared stock deliberately changes the outer draw while holding the two channels fixed.

For general channel matrices U, V and weights p, $J = U\,\mathrm{diag}(p)V^{\mathsf{T}}$. Full column rank on both sides and positive weights give $\mathbf{rank}(J) = q$, even if the charge rule fails. A convention folding two registration rows together can remove a direction from one channel. One can then bound the number of sectors from a rectangular matrix without being able to enumerate all settings available to the stock department.

In topological gravity the weights of the center can depend on a sum over topologies [10]. Here the distribution is supplied with the stock. Under our trace choice, handles multiply numerator and denominator of every normalized moment by the same number. The sewn amplitude evaluates the joint preparation, while the symmetric monoidal functor continues to evaluate separate disks multiplicatively.

7 Conclusion

The declaration of independence applies to the search of the local disposition registers. Extending it to the full certificates requires specifying how the supporting stock was prepared or conditioning on its setting. For an appeal filed solely to establish that the first certificate was unnecessary, the registration of that appeal is still a boundary observable of the batch.

A Handle factors and the vacuum convention

The comultiplication is $\Delta(e_a) = q e_a \otimes e_a$. The handle operator $m \circ \Delta$ has eigenvalue q in each sector. In Eq. (9) the boundary insertions may be evaluated before or after the handles. The empty sphere amplitude is one and the empty torus amplitude is q.

$$Z_{g,n}(f_1, \dots, f_n) = q^g \tau(f_1 \cdots f_n), \qquad \frac{Z_{g,n}(f_1, \dots, f_n)}{Z_{g,n}(1, \dots, 1)} = \tau(f_1 \cdots f_n). \tag{9}$$

For $\tau(e_a) = w_a > 0$, the handle eigenvalue is $1/w_a$ and the unnormalized genus-g weights are w_a^{1-g}. For $n \geq 1$ an alternative is to leave the Frobenius weights uniform and insert the density $a \mapsto q p_a$ on one boundary. Both constructions give the same normalized genus-zero weights for $w_a = p_a$, but only the density insertion keeps the prior p_a at all genera.

Gluing two caps pairs their factors $\tau(e_a)$ with the inverse metric, producing $\sum_a (1/q)^2 q = 1$. Discarding the inverse metric gives $1/q$, which is the agreement probability of independent sector draws. The counting discrepancy would persist even with no visible registrations.

B Integer enumeration

At fixed a, 25 keys output a and one outputs each other row. A cell (x, x) receives $25^2 + 7 = 632$ pairs of keys after summing over sectors. The count in (x, y) for $x \neq y$ is $2(25) + 6 = 56$. In the split case a fixed output has 32 setting-key combinations on either side, which gives 1024 per cell.

In the reference-only control any right output has one key, yielding 32 per cell after summing over the shared sector.

With M keys at each point and k keys per reference row, a nonzero character has unnormalized local sum $(M-8k)\chi_m(a)$. For a neutral n-tuple of nonzero charges the summed moment is $8(M-8k)^n$ before dividing by the dictionary size $8M^n$. Figure 2 uses $M=64$ and $k=8(1-\lambda)$. The $M=32$ and $M=64$ schedules give the same values where they coincide.

To reproduce Figure 1 take the outer product of local count vectors for each a and sum over the sectors. For the conditional calculation, restrict the sum to A_t and divide by $(q/d)32^2$ for each t. Subtract the product of the two normalized conditional marginals to recover Eq. (7).

Acknowledgments

We thank Lisette Vangean for checking the distinction between a returned abstract and an unread backing sheet.

References

[1] Lowell Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *Journal of Knot Theory and Its Ramifications*, 5:569–587, 1996.

[2] Iria Allende Mireis. Independence of concurrent determinations of nonreceipt. Technical Report RMU-24-09, Registry Methods Unit, Brackenholt School of Administration, 2024.

[3] Michael Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988. doi: 10.1007/BF02698547.

[4] Andreas Blommaert, Luca V. Iliesiu, and Jorrit Kruthoff. Gravity factorized. *Journal of High Energy Physics*, 09:080, 2022. doi: 10.1007/JHEP09(2022)080.

[5] Horacio Casini, Marina Huerta, and José Alejandro Rosabal. Remarks on entanglement entropy for gauge fields. *Physical Review D*, 89:085012, 2014. doi: 10.1103/PhysRevD.89.085012.

[6] Vedika N. Chakor. Withdrawal dispositions for absent originating instruments. *Review of Documentary Procedure*, 23:141–153, 2021.

[7] Sidney Coleman. Black holes as red herrings: Topological fluctuations and the loss of quantum coherence. *Nuclear Physics B*, 307:867–882, 1988. doi: 10.1016/0550-3213(88)90110-1.

[8] Robbert Dijkgraaf and Edward Witten. Topological gauge theories and group cohomology. *Communications in Mathematical Physics*, 129:393–429, 1990. doi: 10.1007/BF02096988.

[9] William Donnelly. Decomposition of entanglement entropy in lattice gauge theory. *Physical Review D*, 85:085004, 2012. doi: 10.1103/PhysRevD.85.085004.

[10] John Gardiner and Stathis Megas. 2d TQFTs and baby universes. *Journal of High Energy Physics*, 10:052, 2021. doi: 10.1007/JHEP10(2021)052.

[11] Steven B. Giddings and Andrew Strominger. Loss of incoherence and determination of coupling constants in quantum gravity. *Nuclear Physics B*, 307:854–866, 1988. doi: 10.1016/0550-3213(88)90109-5.

[12] Daniel Harlow and Daniel Jafferis. The factorization problem in Jackiw–Teitelboim gravity. *Journal of High Energy Physics*, 02:177, 2020. doi: 10.1007/JHEP02(2020)177.

[13] Marin Ilescu and Simin E. Navari. Finite support bounds for common preparation across noisy boundaries. *Studies in Discrete Field Theory*, 8:233–250, 2023.

[14] Ilse M. Kerstan and Alain Deyrieux. Registration and abstracting conventions for availability determinations. Technical Report ACRE-22-5, Alderbasin Centre for Records Engineering, 2022.

[15] Joachim Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories*. Cambridge University Press, 2004. doi: 10.1017/CBO9780511615443.

[16] Donald Marolf and Henry Maxfield. Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information. *Journal of High Energy Physics*, 08:044, 2020. doi: 10.1007/JHEP08(2020)044.

[17] Donald Marolf and Jorge E. Santos. AdS Euclidean wormholes. *Classical and Quantum Gravity*, 38:224002, 2021.

[18] Phil Saad, Stephen H. Shenker, and Douglas Stanford. JT gravity as a matrix integral. arXiv preprint arXiv:1903.11115, 2019.

[19] Claude E. Shannon. A mathematical theory of communication. *Bell System Technical Journal*, 27:379–423 and 623–656, 1948.

[20] Anja Tilvis-Kerman and Brice L. Caderel. Record identity in reviews of negative administrative determinations. *Annals of Registry Administration*, 14:91–111, 2025.

[21] Edward Witten. On quantum gauge theories in two dimensions. *Communications in Mathematical Physics*, 141:153–209, 1991.