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Margin Insufficiency in Fermat's Observations on Diophantus: Capacity Estimates from Fourteen Copies and a Test of Reporting Completeness

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Abstract

Fermat's remark that the margin of his copy of Bachet's Diophantus was too narrow to hold a proof of his Last Theorem is the only remark about the margin among the forty-eight observations printed in 1670. We estimate the capacity of the outer margin from 14 copies of the 1621 edition and 31 autograph documents. The mean capacity is 297.5 words (95% CI 274.3 to 322.4), of which 261.5 remain after the note. We compare this with the lengths of the proofs of eight claims of Fermat. None of the earliest published proofs fits. Two of the shortest known proofs fit and six do not. If Fermat mentioned the margin whenever it was too narrow, the expected number of remarks among the 19 observations that assert a result without proof is 14.3, and the observed number is one (binomial P = 2.1e-10; 0.003 at the lower confidence limit for the proportion). Restricted to the four observations in which he claims a demonstration, the count is not unusual (P = 0.051). The proof of Wiles and Taylor-Wiles would occupy 57.6% of the outer margins of the volume.

Keywords: Fermat's Last Theorem, Bachet, Diophantus, marginalia, book history, history of number theory

1 Introduction

Around 1637, according to the tradition that rests on his son's edition of 1670, Pierre de Fermat wrote in the margin of his copy of Bachet's Diophantus that it is impossible to divide a cube into two cubes, or a fourth power into two fourth powers, or in general any power above the second into two of the same kind, and that he had found a truly marvellous proof of this, which the narrowness of the margin would not contain (*hanc marginis exiguitas non caperet*) [1, 3]. The copy itself has not survived. The sentence survives because Samuel de Fermat printed it as Observation II among the forty-eight marginal notes he took from his father's volume.

The historical literature has treated the remark as a statement about mathematics. The question has been whether Fermat had a proof, and if so what it looked like [5, 6, 10]. Most authors who commit themselves doubt that he had a correct one for general exponent. The usual conjecture is that he had an argument of the kind Lamé presented to the Paris Academy in 1847, which

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fails where unique factorisation fails in cyclotomic integers [5]. The general case was settled by Wiles in 1995, in a paper of 109 pages with a further 20 pages by Taylor and Wiles [9, 11].

The remark is also a statement about a physical object. Whether a margin is too small depends on its capacity and on the length of what is to be written in it, and we found no published estimate of the capacity for this volume. A second observation follows from the first. Fermat made many other assertions of general results in the margins of the same book, and most of them have since been proved, at lengths that are on record. If the margin was the limiting factor, those assertions should carry the same remark. They do not. Observation II is the only one that mentions the margin.

This paper has three parts. We estimate the capacity of the outer margin of the 1621 edition at the opening containing Book II, Problem 8, from fourteen surviving copies and a sample of Fermat's autograph hand. We then compare that capacity with the lengths of the proofs of eight of Fermat's claims, using both the earliest published proof and the shortest proof we could find. Finally we test whether the frequency of Fermat's remark on the narrowness of the margin agrees with the frequency with which the margin would have been too narrow. The measurements and the model are described in Section 3, the estimates in Section 4, and their consequences for the historiography in Section 5.

2 Sources

2.1 The edition and the missing copy

Bachet's edition of 1621 prints the Greek text of the *Arithmetica* with a Latin translation in parallel and his own commentary [1]. It is a folio. In Samuel de Fermat's edition of 1670 the observations of his father are printed as a separate series after Bachet's commentary, each keyed to a problem [3], and they are reprinted with the rest of Fermat's surviving writings in the standard edition [8]. The annotated volume is not known to exist. Its margins therefore cannot be measured, and we infer them from other copies.

2.2 Which copies are relevant

Binders trimmed the fore-edge of a book when they sewed and plough-cut it, and rebinders trimmed it again. A margin of 47 mm in 1621 can be 30 mm in 1850. Two things follow for the choice of reference class. First, the note must have fitted inside the margin when it was written, so a later trim could not have limited it, though an earlier trim could. Second, a Toulouse magistrate who annotated a Paris folio in the 1630s was almost certainly annotating a volume in its first binding. We therefore take copies still in contemporary binding as the reference class and use rebound copies only as a check on how much a trimmed fore-edge changes the answer.

2.3 The annotation hand

Fermat's letters and memoranda survive in autograph, and the hand he used for marginalia is smaller and more contracted than the hand of his correspondence. We restricted the handwriting measurements to passages in the small hand, which we identified by an x-height under 1.5 mm in the facsimile plates of the standard edition [8]. Contractions follow the usual practice of seventeenth-century Latin and were expanded according to the standard repertory [2].

3 Materials and methods

3.1 Copies

We obtained measurements for 14 copies of the 1621 edition. Five were measured in person and nine from images with a scale in the frame or from dimensions supplied by the holding library. At the opening containing Book II, Problem 8, we recorded the leaf height and the width of the outer margin, taken from the edge of the leaf to the nearest printed text. The outer margin carries no printed matter at this opening in any of the copies. Each quantity was measured three times and averaged. The standard deviation of repeat measurements was 0.8 mm for copies measured in person and 1.1 mm for copies measured from images. Eight copies are in contemporary binding and six were rebound after 1700. We take the usable height of the margin to be $H = h - 40$ mm, where h is the leaf height, which leaves 20 mm at head and foot.

3.2 Hand

We sampled 31 autograph documents. In each, we took five lines of the small hand and measured the mean horizontal distance per character over ten consecutive characters (p_c, mm) and the mean distance between consecutive baselines (p_ℓ, mm). Document means were used as the unit of analysis. Standard errors in Table 2 are for the mean over documents.

3.3 Contractions and word length

For twelve passages of ten lines each, we transcribed the autograph as written, expanded the contractions, and took the ratio ρ of characters as written to characters in the expanded text. The mean length of a Latin word, including the following space, was taken from a sample of 1,900 words of the 1670 text and is $\ell = 7.0$ characters.

3.4 Capacity

If text is written horizontally in lines across the margin, the number of lines is H / p_ℓ, each line holds w / p_c characters, and a word occupies $\ell \rho$ written characters. Words are not split, so on average half a word is lost at the end of each line. The capacity in words is

$$C_{\parallel} = \frac{H}{p_\ell} \left(\frac{w}{p_c \, \ell \rho} - \frac{1}{2} \right). \tag{1}$$

Text rotated by a quarter turn and written along the length of the margin gives

$$C_{\perp} = \frac{w}{p_\ell} \left(\frac{H}{p_c \, \ell \rho} - \frac{1}{2} \right). \tag{2}$$

Confidence intervals for the mean capacity of a contemporary copy come from 20,000 replicates in which the eight contemporary copies were resampled with replacement and p_c, p_ℓ and ρ were drawn from normal distributions with the means and standard errors in Table 2. Fermat's own copy is one draw from the population of copies, so we also report a prediction interval, in which w and H are drawn from normal distributions with the across-copy means and standard deviations. The wording of the note in Observation II, *Cubum autem in duos cubos* through *non caperet*, is 36 words in the 1670 text. The space left for a proof is therefore $C_r = C_{\parallel} - 36$.

3.5 Claims and proof lengths

We took eight claims attributed to Fermat in the Observations and the correspondence for which a complete proof has since been published [4, 10]. For each we recorded the earliest published proof that is accepted today and the shortest proof we could find in the literature. The length L of a proof is the number of words in a standard English presentation, with each displayed formula counted as eight words and each inline formula as one (Appendix A). For the proof of Wiles and of Taylor and Wiles we used the page count, 129 pages, multiplied by 640 words per page, the median of ten sampled pages counted by the same rule. A proof fits in the margin if $L \le C_r$.

3.6 Reporting of margin insufficiency

Two readers coded the 48 observations of the 1670 edition independently. Set A consists of the observations that assert a general result and give no proof. Set $B \subseteq A$ consists of those in which Fermat also states that he possesses a demonstration. Cohen's κ for membership of A was 0.81, and disagreements were settled by discussion. Let π be the proportion of the eight claims whose shortest known proof does not fit in the margin. If Fermat remarked on the margin whenever the margin was too small, and only then, the number X of observations in a set of size n carrying the remark would follow a binomial distribution with parameters n and π. We report $\Pr(X \le x)$ for the observed count x, with π at its point estimate and at the lower end of its exact 95% interval.

4 Results

4.1 Margins

Table 1 lists the copies. The eight copies in contemporary binding have a mean outer margin of 46.6 mm (SD 2.7) and a mean leaf height of 337.5 mm (SD 2.4). The six later-bound copies have a mean outer margin of 32.3 mm (SD 4.2) and a mean leaf height of 322.3 mm (SD 5.2). The difference in margin width is 14.3 mm (Welch $t = 7.3$, $p < 0.001$). Trimming reduces both dimensions, and it reduces the margin proportionally more (Figure 1). The median page count of the eight contemporary copies is 482, with a range of 478 to 490.

Table 1: Copies of the 1621 edition at the opening containing Book II, Problem 8. Binding: C, contemporary; L, rebound after 1700.

Copy	Binding	Access	Leaf height (mm)	Outer margin (mm)
A	C	Image	338	49
B	C	In person	341	44
C	C	Image	335	51
D	C	Image	336	46
E	C	In person	340	43
F	C	Image	334	48
G	C	Image	337	45
H	C	In person	339	47
I	L	Image	322	33
J	L	Image	318	29
K	L	In person	326	36
L	L	Image	315	27
M	L	Image	324	31
N	L	In person	329	38

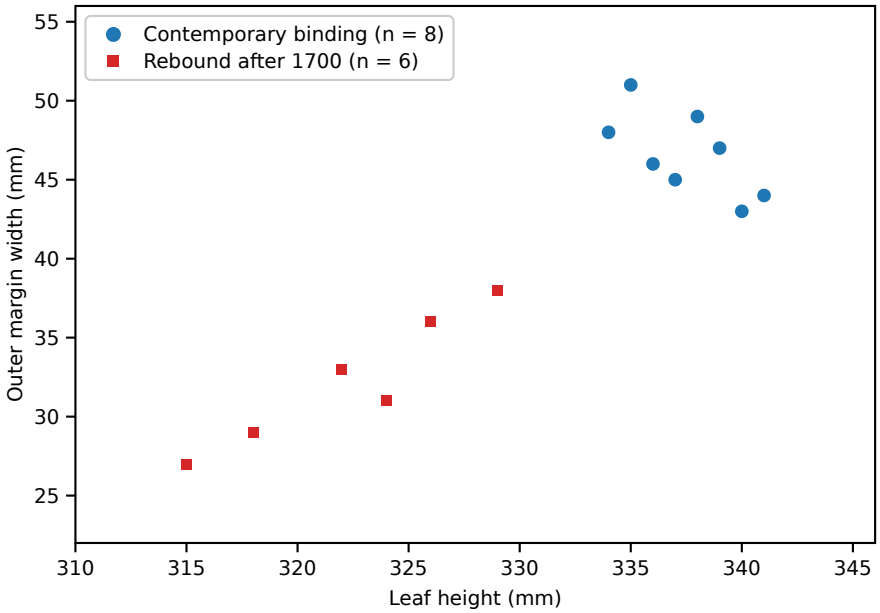


Figure 1: Outer margin width against leaf height for 14 copies of the 1621 edition. Circles are copies in contemporary binding, squares are copies rebound after 1700. Repeat-measurement error is smaller than the markers.

4.2 Capacity of the margin

Table 2 gives the inputs to Eq. (1) and Eq. (2). The usable margin is 46.6 mm wide and 297.5 mm high. In the small hand, Fermat wrote 1.92 mm per character with a baseline distance of 3.4

mm, and contractions saved 11% of the characters of the expanded text.

Table 2: Inputs to the capacity model. Standard deviations are across copies or documents.

Quantity	Symbol	Value	SD	Unit	n
Outer margin width (contemporary copies)	w	46.6	2.7	mm	8
Usable margin height	H	297.5	2.4	mm	8
Character pitch, small hand	p_c	1.92	0.21	mm	31
Baseline distance, small hand	p_ℓ	3.4	0.45	mm	31
Contraction ratio	ρ	0.89	0.04		12
Expanded word length including space	ℓ	7.0		characters	1,900 words

With these inputs, Eq. (1) gives $C_\parallel = \mathbf{297.5}$ words (95% interval 274.3 to 322.4). Rotated text gives $C_\perp = \mathbf{334.4}$ words (310.4 to 360.2). The 95% prediction interval for the horizontal capacity of a single unmeasured contemporary copy is 254.6 to 342.2 words. If the six later-bound copies are used instead, the capacities are 183 and 220 words. The margin holds 0.999 words per millimetre of its length at the mean width. Figure 2 shows how the capacity responds to the two pitch parameters. Across the grid, which runs from 1.6 to 2.3 mm in character pitch and from 2.8 to 4.0 mm in baseline distance, capacity ranges from 205 to 444 words.

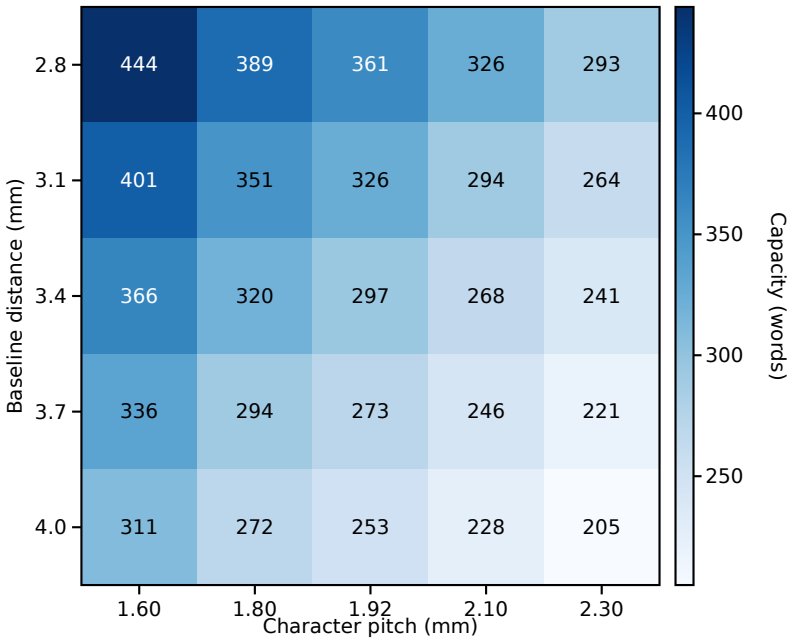


Figure 2: Capacity C of the outer margin, in words, as a function of character pitch and baseline distance, for the horizontal layout of the text. Other inputs as in the table of parameters.

The note itself occupies 36 words, 12.1% of the capacity, which leaves $C_r = \mathbf{261.5}$ words in the horizontal layout and 298.4 words in the rotated layout. We use the horizontal value, which is the smaller, in what follows.

4.3 Lengths of the proofs of Fermat's claims

Table 3 gives the lengths. None of the earliest published proofs fits in C_r. The nearest is Euler's proof of Fermat's little theorem, at 380 words, which is 1.45 times the space available. Two of the shortest known proofs do fit: the textbook proof of Fermat's little theorem, at 70 words, and Zagier's proof that a prime congruent to 1 modulo 4 is a sum of two squares, at 90 words [12]. The other six exceed the margin at every length we found for them, by factors between 1.3 and 316 (Figure 3). The proof of the case $n = 4$ by descent, which Fermat did write down, is 640 words in its original form and 450 in the shortest form we found. It exceeds C_r in both.

Table 3: Fermat's claims, the earliest accepted proof and the shortest proof found, with lengths in words and as multiples of the residual margin $C_r = 261.5$ words. Euler's argument for $n = 3$ contained a gap that was repaired later [5]; the length given is for the repaired argument.

Claim	Earliest proof	L_e	L_e/C_r	Shortest proof	L_s	L_s/C_r
$a^{p-1} \equiv 1 \pmod{p}$	Euler, 1736	380	1.45	Textbook	70	0.27
Prime $p \equiv 1 \pmod{4}$ is $x^2 + y^2$	Euler, 1749	2,900	11.09	Zagier, 1990	90	0.34
Every integer is a sum of four squares	Lagrange, 1770	2,200	8.41	Hurwitz, 1896	550	2.10
$x^2 - Dy^2 = 1$ is solvable	Lagrange, 1768	3,100	11.85	Pigeonhole argument	340	1.30
Polygonal number theorem	Cauchy, 1813	4,000	15.30	Nathanson, 1987	1,600	6.12
$x^4 + y^4 = z^2$ has no solution	Fermat, c. 1640	640	2.45	Textbook	450	1.72
$x^3 + y^3 = z^3$ has no solution	Euler, 1770	1,900	7.27	Textbook	900	3.44
$x^n + y^n = z^n$, $n \geq 3$	Wiles, Taylor–Wiles, 1995	82,560	315.72	Same	82,560	315.72

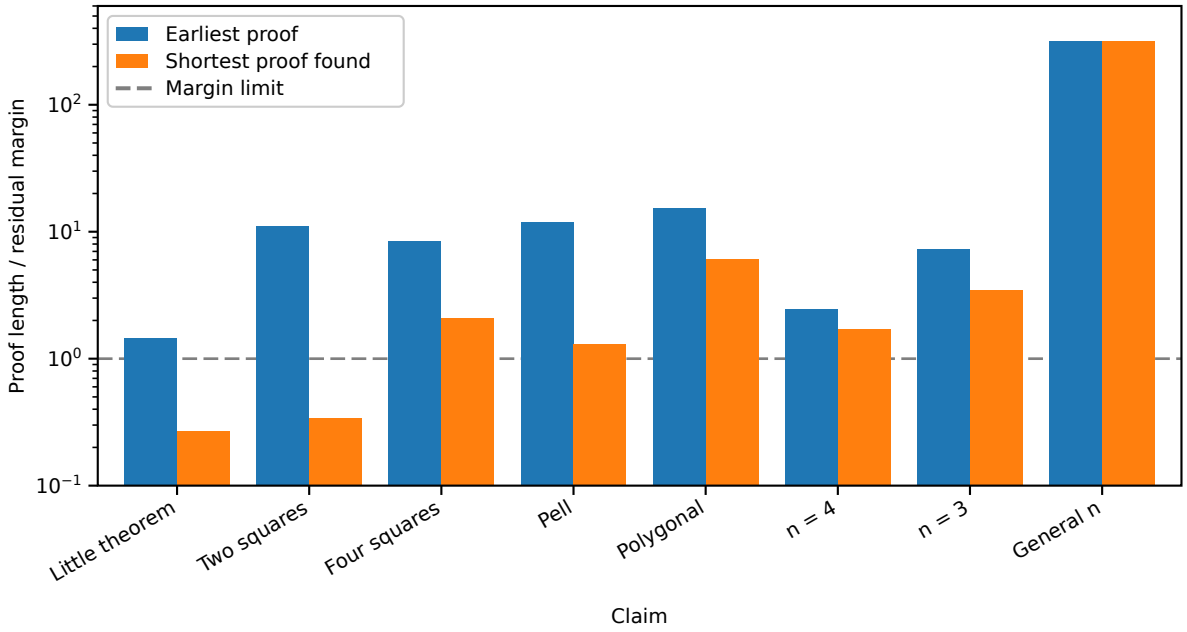


Figure 3: Length of proof divided by the residual margin, for the earliest accepted proof and the shortest proof found, for each of eight claims of Fermat. The dashed line is a ratio of one. Proofs below it fit in the margin.

4.4 Reporting of margin insufficiency

Of the 48 observations, 19 are in set A and 4 are in set B. Observation II is in both sets, and it is the only observation in either that mentions the margin. Six of the eight shortest known proofs exceed C_r, so $\hat{\pi} = 0.75$ (exact 95% interval 0.35 to 0.97). If the margin had been mentioned whenever it was too narrow, we would expect 14.3 of the 19 observations in A to carry the remark, against one observed. The probability of one or none is 2.1×10^{-10} at $\hat{\pi} = 0.75$ and 3.2×10^{-3} at $\pi = 0.35$. In set B the expected count is 3.0 of 4, and the probability of one or none is 0.051 at $\hat{\pi}$ and 0.56 at the lower limit. With the earliest published proofs in place of the shortest, $\pi = 1$ and the probability for set A is zero.

4.5 Scale

The proof of Wiles and Taylor–Wiles has an estimated length of 82,560 words. At the measured density it needs 82.6 m of margin 46.6 mm wide, or equivalently a single margin 12.9 m wide on a leaf 297.5 mm high. The outer margins of the 482 pages of the volume have a total length of 143.4 m, so the proof would take 57.6% of them.

5 Discussion

For the nineteen unproved assertions the data do not agree with the hypothesis that Fermat remarked on the margin whenever it was too narrow. The probability of the observed count is small even at the least favourable π in the interval. We read this as under-reporting. On the evidence of later proof lengths the margin would have been insufficient for at least six of his claims, and he recorded the insufficiency once.

There are two ways to avoid the conclusion. The first is to restrict the test to set B, the four observations in which Fermat says he has a demonstration. A remark about the narrowness of a margin makes sense only for a proof one has, and in set B the count of one is not unusual (0.051 at $\hat{\pi}$, 0.56 at the lower limit). We find this the more natural reading, though it leaves three observations in B in which Fermat claimed a proof that, on our estimates, would probably not have fitted, and said nothing. The second is to suppose that for the other assertions Fermat did not have a proof, and had no occasion to complain about the space. The data cannot separate these accounts from under-reporting. They place a bound on how many observations in A he could have mentioned.

Two of the shortest known proofs fit in the margin. One of them, the proof of the two-squares theorem, was published in 1990, and is shorter than Euler's by a factor of 32. The other is the textbook proof of the little theorem. A margin of this size could have held a proof of either result, so a remark about the margin would not have been expected for them, and none is recorded. These two claims are therefore consistent with the test whichever way it is read.

The proof for $n = 4$ sits awkwardly with the capacity estimate. At 640 words it is longer than two margins, yet Samuel de Fermat printed it as a single observation [3]. Either the descent was written across several consecutive margins or onto a blank leaf, or it was written at a place with a wider margin than the opening we measured, or it was composed more densely than the small hand we sampled. We did not identify the opening in any of the 14 copies. If the third explanation holds, the character pitch used here is too large, and the first column of Figure 2 (1.6 mm, a capacity of 366 words) is the more appropriate one. At 1.6 mm the residual margin is 330 words, and six of the eight shortest proofs still exceed it. A capacity above 376 words, which is within the grid, would bring the shortest proof for Pell's equation (340 words) inside the margin and lower $\hat{\pi}$ to 0.625.

A rhetorical reading of the phrase is possible, in which *hanc marginis exiguitas non caperet* is a conventional way of declining to continue and the margin plays no role. This reading predicts no relation between the margin and the length of the proof, and nothing in our data can test it. We have assumed throughout that the statement is literal. A related case is the note in the manuscript of Galois, *Je n'ai pas le temps* [7], which gives a temporal rather than a spatial limit. The analogue of Eq. (1) would need a writing speed, and we have no data on it.

The limitations are these. The fourteen copies are those we could reach, and they are not a random sample of the edition. We measured one opening, and the outer margin at Book II, Problem 8 may not be representative. Fermat's copy might have been bound with blank leaves that we have not modelled. Eight claims is a small set, so $\hat{\pi}$ is imprecise. The word counts in Table 3 are our own and depend on the presentation chosen. The count of 640 words per page for the Annals papers is a median over ten pages, and the formulae there are denser than in the classical proofs, so the length for the proof of 1995 is probably an underestimate in words.

6 Concluding remarks

The uncertainty in $C_{\|}$ for a single copy has two parts. Copy-to-copy variation in margin width accounts for about two thirds of the variance of the prediction, and the character pitch, baseline distance and contraction ratio account for the rest. The fore-edge width of Fermat's own copy would be the single most useful measurement, and it is the only one that cannot be obtained. With it, the 95% prediction interval for $C_{\|}$ would narrow from 88 words to about 51. The classification of the observations is the other open item. A third reader coding set B would settle

whether it contains four observations or five, and with five the probability in the last section changes from 0.051 to 0.016 at $\hat{\pi}$.

Data availability

Measurements for each of the 14 copies, the handwriting samples, the coding of the 48 observations and the code for Eq. (1) and the test in Section 3 are available from the corresponding author.

A Counting words in proofs

The word counts in Table 3 were made on the following rule. Prose was counted in words of English. A displayed formula was counted as eight words, which is the median number of tokens in the displayed formulae of ten sampled pages of the Annals paper, rounded. Inline formulae were counted as one word each. Diagrams were not counted. For the proofs of Euler, Lagrange, Cauchy and Fermat the counts are of the presentation in the book of Edwards or the history of Weil where one is given there [5, 10], and otherwise of a translation of the original. Rounding is to the nearest ten for lengths below 1,000 words and to the nearest hundred above, except for the final row, which is the product of 129 pages and 640 words.

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